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THE SOLUTION OF BOUNDARY-VALUE PROBLEMS OF THE LONGITUDINAL-TRANSVERSE BENDING OF ORTHOTROPIC CIRCULAR PLATES ON A LINEARLY ELASTIC BASE†

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A technique is developed for solving boundary-value problems of non-axisymmetric longitudinal-transverse bending in thin cylindrical orthotropic linearly thermo-elastic annular and solid circular plates on a linearly elastic base in a classical setting. The given and unknown functions are represented by Fourier expansions. The solving system of fourth-order ordinary differential equations is of Bessel type. The solution of the homogeneous system is obtained by a technique developed previously‡ (which generalises the Neumann–Weber–Schläfli technique [1–5]) for determining fundamental solutions in the form of generalized power series – higher-order cylindrical functions of first, second and higher kinds, based on the property of the continuous dependence of the solutions on the parameters. Particular solutions are determined by Lagrange's method (the variation of arbitrary constants). The results of numerical calculations are presented for plates with a hinge-supported outer contour in which the inner contour is loaded with distributed bending moments. © 2002 Elsevier Science Ltd. All rights reserved.

Within the framework of the model of cylindrically orthotropic circular plates on a linearly elastic base (see monographs and surveys [6–10]), exact solutions have been obtained for axisymmetric bending [6, 7] and for special cases [8–10]. No solutions of general form are available for problems of longitudinal-transverse non-axisymmetric bending. Such a solution will be presented below.

1. BASIC ASSUMPTIONS AND EQUATIONS OF THE PROBLEM

Consider a circular (annular or solid) thin plate of constant thickness h, outer radius $r = a$ and inner radius $r = b$, in a polar system of coordinates. The plate is attached to a linearly elastic base (in Winkler's sense), whose coefficients of resistance in the radial, circumferential and transverse directions, K_u, K_v and K_w, respectively, are constant. The plate is subject to distributed loads: radial $q_1(r, \theta)$, tangential $q_2(r, \theta)$ and normal $q_z(r, \theta)$, reduced to the middle surface of the plate, and is heated from an initial temperature $T_0(r, \theta, z)$ in the natural state to a temperature $T(r, \theta, z)$. The linearly thermo-elastic (Hooke–Duhamel–Neumann) deformations of the plate are small (in Cauchy's sense), and the geometrical Kirchhoff relations are satisfied, as are the conditions for the generalized plane stressed state. The principal axes of cylindrical orthotropism coincide with a cylindrical system of coordinates attached to the middle (base) surface. Surfaces equidistant from the middle surface are bent similarly, so that their Lamé parameters and radii of curvature coincide. Inner layers of the plate do not affect one another. The effect of longitudinal stresses on the bending of the plate is negligibly small. The outer loads distributed over the plane and over the contours (specific radial stresses $N_{1b}(\theta)$, $N_{1a}(\theta)$, transverse stresses $R_{1b}(\theta)$, $R_{1a}(\theta)$, and bending moments $M_{1b}(\theta)$, $M_{1a}(\theta)$, reduced to the middle plane), the temperature distribution or displacements (radial $u_b(\theta)$, $u_a(\theta)$, circumferential $v_b(\theta)$, $v_a(\theta)$, and deflection $w_b(\theta)$, $w_a(\theta)$) and their derivatives and linear combinations are represented by Fourier series in the circular coordinate θ.

It is required to determine the values of the radial and tangential displacements, $u(r, \theta)$ and $v(r, \theta)$, of the middle (base) plane; its deflection $w(r, \theta)$, specific radial and circumferential stresses $N_1(r, \theta)$ and $N_2(r, \theta)$, and shear $S(r, \theta)$; the specific radial and circumferential bending moment, $M_1(r, \theta)$ and $M_2(r, \theta)$, and the torque $H(r, \theta)$, the specific transverse radial stress $Q_1(r, \theta)$ and circumferential stress

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$Q_2(r, \theta)$, which are statically equivalent, respectively, to the inner stresses and moments of the inner stresses about local coordinate axes, where the latter are the tangents to the coordinate grid of a global (cylindrical) system of coordinates (r, θ, z) with origin at the centre of the circumference of the inner contour of the plate and z axis directed along the normal to the middle plane.

The equations of small longitudinal-transverse bending of cylindrically orthotropic thin elastic plates, the unknown functions being the radial and circumferential (tangential) displacements $\bar{u}(p, \theta) = u/a$ and $\bar{v}(p, \theta) = v/a$, in units of the radius a, and the deflection $\bar{w}(p, \theta) = w/h$ of the middle surface, in units of the thickness h, have the following form [11, 12] (henceforth the bars over $\bar{u}$ $\bar{v}$, $\bar{w}$ and over the dimensionless relative coordinate $\bar{z} = z/h$ will be omitted)

equilibrium in the radial and circumferential directions (a coupled system of two equations)

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} - \frac{c}{\rho^2}u + \frac{d}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + (\omega_{21} + d)\frac{1}{\rho}\frac{\partial}{\partial \theta}\left[\frac{\partial v}{\partial \rho} - \frac{c+d}{\omega_{21}+d}\frac{v}{\rho}\right] = f_1(\rho,\theta) - k_u^2 u \tag{1.1}$$

$$\frac{\omega_{21}+d}{d}\frac{1}{\rho}\frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial \rho} + \frac{c+d}{\omega_{21}+d}\frac{u}{\rho}\right] + \frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} - \frac{v}{\rho^2} + \frac{1}{\rho}\frac{c}{d}\frac{\partial^2 v}{\partial \theta^2} = f_2(\rho,\theta) - k_v^2 v \tag{1.2}$$

equilibrium relative to bending (one decoupled equation)

$$\frac{\partial^4 w}{\partial \rho^4} + \frac{2}{\rho}\frac{\partial^3 w}{\partial \rho^3} - \frac{c}{\rho^2}\frac{\partial^2 w}{\partial \rho^2} + \frac{c}{\rho^3}\frac{\partial w}{\partial \rho} + \frac{c}{\rho^4}\frac{\partial^4 w}{\partial \theta^4} + \frac{2(\omega_{21}+2d)}{\rho^2}\frac{\partial^4 w}{\partial \rho^2 \partial \theta^2} -$$

$$-\frac{2(\omega_{21}+2d)}{\rho^3}\frac{\partial^3 w}{\partial \rho \partial \theta^2} + \frac{2(c+\omega_{21}+2d)}{\rho^4}\frac{\partial^2 w}{\partial \theta^2} = f_z(\rho,\theta) - k_w^4 w \tag{1.3}$$

where

$$c = \frac{E_2}{E_1} = \frac{B_2}{B_1} = \frac{D_2}{D_1}; \quad B_i = \frac{E_i h}{1-\omega_{21}\omega_{12}}, \quad D_i = \frac{E_i h^3}{12(1-\omega_{12}\omega_{21})}, \quad i = 1, 2$$

$$2d = \frac{2G_{12}(1-\omega_{21}\omega_{12})}{E_1} = \frac{B_3}{B_1} = \frac{D_3}{D_1}$$

$$e = \omega_{21} + 2d, \quad B_3 = G_{12}h, \quad D_3 = \frac{2G_{12}h^3}{12}$$

$$k_u^2 = \frac{K_u a^2}{B_1 h}, \quad k_v^2 = \frac{K_v a^2}{B_3 h}, \quad k_w^4 = \frac{K_w a^4}{D_1 h}$$

$$f_1(\rho,\theta) = \frac{q_1(\rho,\theta)a^2}{B_1} + q_{1T}(\rho,\theta), \quad f_2(\rho,\theta) = \frac{q_2(\rho,\theta)a^2}{B_3} + q_{2T}(\rho,\theta)$$

$$f_z(\rho,\theta) = \frac{q_z(\rho,\theta)a^4}{D_1 h} - q_{zT}(\rho,\theta)$$

$$q_{1T}(\rho,\theta) = (\alpha_1 + \omega_{21}\alpha_2)\frac{\partial n_T}{\partial \rho} + (\alpha_1(1-\omega_{21}) + (\omega_{21} - c)\alpha_2)\frac{n_T}{\rho}$$

$$q_{2T}(\rho,\theta) = \frac{1}{d}\left[(\omega_{21}\alpha_1 + c\alpha_2)\frac{1}{\rho}\frac{\partial n_T}{\partial \theta}\right]$$

$$q_{zT}(\rho,\theta) = \frac{a^2}{h^2}\left\{(\alpha_1 + \omega_{21}\alpha_2)\frac{\partial^2 m_T}{\partial \rho^2} + ((\alpha_1 + \omega_{21}\alpha_2) - c(\omega_{12}\alpha_1 + \alpha_2))\frac{1}{\rho}\frac{\partial m_T}{\partial \rho} + \right.$$

$$+c(\omega_{12}\alpha_1+\alpha_2)\frac{1}{\rho^2}\frac{\partial^2 m_T}{\partial\theta^2}\Bigg\}$$

$$n_T=\int_{-1/2}^{1/2}t(\rho,\theta,z)dz,\quad m_T=12\int_{-1/2}^{1/2}t(\rho,\theta,z)zdz$$

$$t(\rho,\theta,z)=T(\rho,\theta,z)-T_0(\rho,\theta,z)$$

$\rho = r/a$ is the dimensionless radial coordinate, $\rho \leqslant 1$; z is the relative thickness coordinate, $-1/2 \leqslant z \leqslant 1/2$; c is the coefficient of orthotropism; E_1 and E_2 are the Young's moduli in the radial and circumferential directions; B_i are the tensile and compressive stiffnesses; D_i are the bending stiffnesses; $2d$ is the coefficient of shear orthotropism; e is the reduced coefficient of shear orthotropism; ω_{21} and ω_{12} are the coefficients of transverse strain in the direction 2 (or 1) for expansion (compression) in the direction 1 (or 2) (Poisson's ratios); B_3 is the shear stiffness; D_3 is the torsion stiffness; k_u^2, k_v^2 and k_w^2 are the radial, circumferential and lateral reduced "bed" coefficients, respectively; $f_1(\rho, \theta), f_2(\rho, \theta)$ and $f_z(\rho, \theta)$ are the radial, tangential and normal generalized "force" loads, respectively; $q_1(\rho, \theta)$, $q_2(\rho, \theta)$ and $q_z(\rho, \theta)$ are the given radial, circumferential and normal distributed force loads, respectively; $q_{1T}(\rho, \theta)$, $q_{2T}(\rho, \theta)$ and $q_{zT}(\rho, \theta)$ are the given radial, circumferential and normal distributed "thermal loads," respectively; α_1 and α_2 are the coefficients of linear thermal expansion in the radial and circumferential directions; n_T is the average temperature and m_T is the average temperature gradient over the plate thickness; $t(\rho, \theta, z)$ is the difference between the actual temperature and the natural temperature $T_0(\rho, \theta, z)$ of the unstressed and unstrained state.

The following relations hold for the elastic moduli of an orthotropic body, Poisson's ratios and the coefficients of linear thermal expansion

$$E_1\omega_{21}=E_2\omega_{12},\quad E_1\alpha_1^2=E_2\alpha_2^2$$

The last relation follows from an analysis of the thermodynamic potentials (the internal energy $U(\varepsilon, q)$, the free energy $F(\varepsilon, T)$, Gibbs potential $X(\sigma, T)$ and the enthalpy $Y(\sigma, q)$) [13].

2. THE SOLVING SYSTEM OF EQUATIONS

Let us represent the known loads and temperatures

$$\{u_1, u_\beta, \vartheta_1, \vartheta_\beta, n_{11}, n_{1\beta}, m_{11}, m_{1\beta}, s_{1\beta}, s_{11}, r_{11}, r_{1\beta}\}$$

distributed over the plane, the inner contour $\rho = \beta = b/a$ and the outer contour $\rho = 1$, and also the unknown functions

$$\{u, v, w, n_1, n_2, s_1, m_1, m_2, h_1, r_1, r_2\}$$

where the force characteristics are divided by the stiffnesses

$$n_{(1,2)}=\frac{N_{(1,2)}}{B_1},\quad m_{(1,2)}=\frac{M_{(1,2)}a^2}{D_1h},\quad r_{(1,2)}=\frac{R_{(1,2)}a^3}{D_1h},\quad h_1=\frac{Ha^3}{D_3h},\quad s_1=\frac{S}{B_3}$$

by the Fourier expansions

$$(u,v,w)(\rho,\theta)=\frac{1}{2}(u,v,w)_0(\rho)+\sum_{p=1}^{\infty}[(u,v,w)_{cp}(\rho)\cos p\theta+(u,v,w)_{sp}(\rho)\sin p\theta] \tag{2.1}$$

where the functions $(u, v, w)_{cp}(r)$ and $(u, v, w)_{sp}(r)$ – the coefficients of the cosines (symbol c) and sines (symbol s) – depend on the harmonic number p. The initial system of partial differential equations is then reduced to a system of ordinary differential equations of the eighth (fourth and fourth) order for one group of components, e.g. for the cosines:

$$B_{11}^{(2)}\{u_{cp}\}+\lambda_{12}^{s}E_{12}^{(1)}\{v_{sp}\}=f_{1cp}(\rho)\rho^2$$

$$\lambda_{21}^{c}E_{21}^{(1)}\{u_{cp}\}+B_{22}^{(2)}\{v_{sp}\}=f_{2sp}(\rho)\rho^2 \tag{2.2}$$

$$B_{33}^{(4)}\{w_{cp}\}=f_{zcp}(\rho)\rho^4$$

and an analogous system for the other (sines), differing only in the signs of the coefficients $\lambda_{12}^{s}=\lambda_{12}^{c}$ and $\lambda_{21}^{s}=\lambda_{21}^{c}$.

Here we have used the following notation for differential operators of order L_{ij} of Euler and Bessel types

$$E_{ij}^{(L_{ij})}\{y\}=\prod_{l=1}^{L_{ij}}(D-\nu_{ijl})\{y\},\quad B_{ij}^{(L_{ij})}\{y\}=E_{ij}^{(L_{ij})}\{y\}+k_y^{L_{ij}}\rho^{Z_{ij}}y$$

expressed in polynomial form as the commutative product of elementary binomial Euler operators $E_{ij}^{(1)}\{y\}=(D-\nu_{ij})\{y\}$, where

$$D\{y\}=\rho\frac{dy}{d\rho},\quad D^{(n)}\{y\}\equiv\underbrace{\rho\frac{d}{d\rho}\left\{\rho\frac{d}{d\rho}\left[\dots\left(\rho\frac{dy}{d\rho}\right)\right]\right\}}_{n\text{ times}}$$

is a differential operator in the system of polar coordinates

$$\nu_{11(1,2)}=\pm(c+dp^2)^{1/2},\quad \nu_{121}=-\nu_{211}=\frac{c+d}{\omega_{21}+d}$$

$$\nu_{22(1,2)}=\pm\left(\frac{d+cp^2}{d}\right)^{1/2},\quad \nu_{33(1,2)}=1\pm\lambda_{331},\quad \nu_{33(3,4)}=1\pm\lambda_{332}$$

$$\lambda_{33(1,2)}=\left[\frac{1+A_3}{2}\pm\left[\left(\frac{1-A_3}{2}\right)^2-B_3^*\right]^{1/2}\right]^{1/2}$$

are the characteristic parameters ν_{ijl} of the binomial operators

$$A_3=c+2ep^2,\quad B_3^*=[cp^2-2(c+e)]p^2,\quad \lambda_{12}^{s,c}=\pm p(\omega_{21}+d),\quad \lambda_{21}^{c,s}=\mp p\frac{\omega_{21}+d}{d}$$

are the coefficients of the system, and Z_{ij} is the exponent of the power of the Bessel correction.

3. SOLUTION OF THE SYSTEM

The solution of system (2.2), consisting of the general solution $\{\bar{u},\bar{v},\bar{w}\}$ of the homogeneous system, expressed in terms of fundamental solutions $\{\bar{u}_1(\rho),\bar{v}_1(\rho),\bar{w}_1(\rho)\}$ and arbitrary constants A_l, and a particular solution $u(\rho)$, $v(\rho)$, $w(\rho)$ of the inhomogeneous system

$$u_{cp}(\rho)=\sum_{l=1}^{4}A_{cpl}\bar{u}_{cpl}(\rho)+U_{cp}(\rho),\quad v_{sp}(\rho)=\sum_{l=1}^{4}A_{cpl}\bar{v}_{cpl}(\rho)+V_{sp}(\rho)$$

$$w_{cp}(\rho)=\sum_{l=1}^{4}B_{cpl}\bar{w}_{cpl}(\rho)+W_{cp}(\rho), \tag{3.1}$$

as a system of Bessel type, will be expressed as generalized power series [9]

$$y_l(\rho)=\rho^{\nu_l}\left(C_{yl}^{(0)}+\sum_{m=1}^{\infty}C_{yl}^{(m)}\rho^m\right),\quad y_l(\rho)=\{(u,v,w)_{cpl}(\rho),(u,v,w)_{spl}(\rho)\} \tag{3.2}$$

The characteristic exponents ν_l are determined from the appropriate characteristic equations.

The system of fundamental solutions of the two coupled second-order equations and one decoupled fourth-order equation of Bessel type, in accordance with formula (3.2), contains numbers ν_l which are the roots of the characteristic (secular) equations. The secular equations for the systems specified above are formed as follows.

For the fourth-order differential equation (2.2), which is the sum of a fourth-order Euler operator $E_{33}^{(4)}\{w\}$ and the Bessel correction $k_w^4\rho^4 w$, the secular equation is determined by the characteristic parameters of the Euler operator, and in this case we have a biquadratic equation

$$P_3^{(4)}(\nu)=(\nu-1)^4-(1+A_3)(\nu-1)^2+(A_3+B_3^*)=0, \tag{3.3}$$

whose roots ν_n are identical with the exponents $\nu_{33l}(l = 1, 2, 3, 4)$.

We will assume that the numbering of the roots or parameters ν_{33l} is such that their values for $c \geqslant 1$ are arranged in decreasing order with increasing value of the subscript l. Note that for $p = 1$ and arbitrary c (when $\lambda_{331} = 1 + (c + 2e)^{1/2}$ and $\lambda_{332} = 0$) the two roots $\nu_{332} = \nu_{333} = 1$ are identical, and for $p = 0$ and $c = 1$ (when $\lambda_{332} = \lambda_{333} = 1$) there are two identical pairs of roots: $\nu_{332} = \nu_{333} = 2$ and $\nu_{333} = \nu_{334} = 0$. The first case is that of bending according to the first harmonic, and the second is that of axisymmetric bending modes of an isotropic circular plate.

The characteristic equation for system (2.2) of two differential equations of Bessel type is obtained from the determinant of the corresponding Euler-type equation (when only complete Euler operators remain on the left-hand side) by the substitution $D \Rightarrow \nu$. We have

$$P_4^{(4)}(\nu)=\det\begin{Vmatrix} P_{11}^{(2)}(\nu) & \lambda_{12}^s P_{12}^{(1)}(\nu) \\ \lambda_{21}^c P_{21}^{(1)}(\nu) & P_{22}^{(2)}(\nu)\end{Vmatrix}=0$$

$$E_{ij}^{(L_{ij})}\{y\}=\prod_{l=1}^{L_{ij}}(D-\nu_{ijl})\{y\}\Rightarrow P_{ij}^{(L_{ij})}(\nu)=\prod_{l=1}^{L_{ij}}(\nu-\nu_{ijl})$$

As a result we again obtain a biquadratic characteristic equation

$$P_4^{(4)}(\nu)=\nu^4-A_4\nu^2+B_4=0 \tag{3.4}$$

where the characteristic roots (exponents) ν_{44l} are

$$\nu_{44(1,4)}=\pm\lambda_{441},\quad \nu_{44(2,3)}=\pm\lambda_{442},\quad \lambda_{44(1,2)}=\{A_4/2\pm[(A_4/2)^2-B_4]^{1/2}\}^{1/2}$$
$$A_4=(c+dp^2)(1+1/d)+p^2(\omega_{21}+d)^2/d,\quad B_4=(c+dp^2)^2/d-p^2(c+d)^2/d$$

When $p \geqslant 2$ the characteristic roots are simple (non-multiple, and the difference of any two is not a multiple of two). For $p = 1$ and arbitrary c (when $\lambda_{441} = c + d)(1 + 1/d) + (\omega_{21} + d)^2/d, \lambda_{442} = 0$) we have two identical characteristic numbers $\nu_{442} = \nu_{443} = 0$, while for $p = 0$ (when $\lambda_{441} = \sqrt{c}$ and $\lambda_{442} = \sqrt{c/d}$) the system splits into two decoupled equations with independent characteristic numbers.

The fundamental solutions depend on the multiplicity of the roots, in particular, solutions corresponding to non-multiple roots are simple, in the form of generalized power series, while those corresponding to multiple roots are generalized power series in which the powers of the logarithms are different from the multiplicity exponent of the roots.

4. CLASSIFICATION OF THE SOLUTIONS. MULTIPLIERS OF CHARACTERISTIC ROOTS

We will classify the roots of Nth order equations by the technique described in the paper cited in the footnote on page 963. We introduce an anti-symmetric matrix of multipliers, whose components are the differences of pairs of characteristic numbers (roots, exponents), arranged in decreasing order and divided by the power exponent Z_{ij} of the Bessel correction. The components of the multipliers that satisfy the relations.

$$\mu_{ijkl}=\frac{\nu_{ijk}-\nu_{ijl}}{Z_{ij}},\quad \mu_{ijkl}=-\mu_{ijmk}+\mu_{ijml},\quad k,l,m=1,2,3,4$$
$$\nu_{ijk}\geqslant\nu_{ijl}\geqslant\nu_{ijm}\quad \text{for}\quad m\geqslant\kappa\geqslant l \tag{4.1}$$

in one row form an increasing sequence. For example, for the roots of the first secular equation (when $Z_{33}=4$ is identical with the order of the equation) and of the second (when $Z_{44}=2$ is not the same as the order of the system), the elements of the multiplier matrix are as follows:

$$\mu_{33kl}=\frac{\nu_{33k}-\nu_{33l}}{4},\quad \mu_{44kl}=\frac{\nu_{44k}-\nu_{44l}}{2},\quad k,l=1,2,3,4$$

The over-diagonal components of the matrix $|\mu_{33kl}|$ are always positive, the diagonal ones are zero, and the subdiagonal ones are negative.

The denominators of coefficients of series (3.2) contain the products

$$\prod_{k=1}^{4}\prod_{\xi=i}^{m}(\mu_{ijkl}+\xi)$$

whose factors are sums of multipliers μ_{ijkl} and the natural numbers $\xi=m\in N$ which vanish when the multipliers take negative integer values. We distinguish among the rows of multipliers: simple rows contain only fractional subdiagonal components, and singular rows contain negative integer-valued or zero subdiagonal components. The kind (or multiplicity) of a row is determined by the number of its singular elements. Two rows – the kth, simple, row and the lth, singular, row – are said to be conjugate with respect to the multiplier

$$\mu_{ijkl}=-\mu_{ijkl}=m_{ijkl}=0,\ 1,\ 2,\ldots$$

The first (simple) row is said to be basic generating, and the second, conjugate-generating. Two rows, the first basic generating row and the first singular row of the first kind which has only one singular element, form a root pair. All multiple solutions of second, third, etc. kinds can be expressed in terms of root pairs. There may be several root pairs, with an appropriate number of kinds for each. To a simple row there corresponds a solution of the first kind and to a singular row solutions of the second, third, etc., kins. For the systems being considered here, there are most frequently solutions of the first and second kinds, more rarely also of the third kind.

5. SOLUTIONS OF THE FIRST KIND

Using a standard procedure [1–4, 14, 15] to determine the expansion coefficients of the generalized power series (3.2) for the case of simple multipliers, we express solutions of the first kind as follows:

for the deflection $\bar{w}^{(1)}_{cpn}$ $(n=1,2,3,4)$

$$\bar{w}^{(1)}_{cpn}=\rho^{\nu_{33n}}\Sigma^{(4)}_{33n}(\rho)$$
$$\Sigma^{(L_{ij})}_{ijn}(\rho)=\left\{Z_{ij}\prod_{k=1,k\neq n}^{L_{ij}}(\mu_{ijkn})\right\}^{-1}+\sum_{m=1}^{\infty}(-1)^m\left(\frac{k_w\rho}{Z_{ij}}\right)^{L_{ijm}}\left\{\prod_{k=1}^{L_{ij}}\prod_{\xi=1}^{m}(\mu_{ijkn}+\xi)\right\}^{-1} \tag{5.1}$$

for the displacements $\bar{u}^{(1)}_{cpn}$ and $\bar{v}^{(1)}_{spn}$ $(n=1,2,3.\ 4)$

$$\bar{u}^{(1)}_{cpn}=\rho^{\nu_{44n}}\left(C^{(0)}_{u,n}+\sum_{m=1}^{\infty}C^{(m)}_{u,n}\rho^{2m}\right),\quad \bar{v}^{(1)}_{spn}=\rho^{\nu_{44n}}\left(C^{(0)}_{v,n}+\sum_{m=1}^{\infty}C^{(m)}_{v,n}\rho^{2m}\right) \tag{5.2}$$

The coefficients $C^{(m)}_{u,n}$ and $C^{(m)}_{v,n}$ $(m\geqslant1)$ are given by the recurrence relations

$$\left\|C^{(m)}_{i,n}\right\|=\left\|\alpha^{(m)}_{ij,n}\right\|\left\|C^{(m-1)}_{j,n}\right\|=\prod_{k=1}^{m}\left\|\alpha^{(k)}_{ij,n}\right\|\left\|C^{(0)}_{j,n}\right\|,\quad i,j=u,v$$

or

$$C_{u,n}^{(m)} = \alpha_{11,n}^{(m)} C_{u,n}^{(m-1)} + \alpha_{12,n}^{(m)} C_{v,n}^{(m-1)}, \quad C_{v,n}^{(m)} = \alpha_{21,n}^{(m)} C_{u,n}^{(m-1)} + \alpha_{22,n}^{(m)} C_{v,n}^{(m-1)}$$

$$C_{u,n}^{(0)} = C_{u,n}^{(0)}, \quad C_{v,n}^{(0)} = \frac{P_{11}^{(2)}(\nu_{44n})}{\lambda_{12}^{s} P_{12}^{(1)}(\nu_{44n})} C_{u,n}^{(0)} \tag{5.3}$$

with coefficients matrix $\|\alpha_{ij,n}^{(m)}\|$ defined by

$$\alpha_{11,n}^{(m)} = -k_u^2 \tilde{P}_{11}^{(2)}, \quad \alpha_{12,n}^{(m)} = \lambda_{12}^{s} k_v^2 \tilde{P}_{12}^{(1)},$$

$$\alpha_{21,n}^{(m)} = \lambda_{21}^{s} k_u^2 \tilde{P}_{21}^{(1)}, \quad \alpha_{22,n}^{(m)} = -k_v^2 \tilde{P}_{22}^{(1)}$$

$$\tilde{P}_{ij}^{(k)} = \frac{P_{ij}^{(k)}(\nu_{44n} + 2m)}{P_{44}^{(4)}(\nu_{44n} + 2m)}, \quad i,j,k = 1,2$$

The four arbitrary integration constants $A_{cpn} = C_{1,n}^{(0)}$ for longitudinal displacements and the four constants B_{cpn} for lateral displacements (n = 1, 2, 3, 4) are determined from the boundary conditions.

For integer multipliers $\mu_{33nk} = -\mu_{33kn} = m_{33nk}$, solutions of the first kind $\tilde{w}_{cpn}^{(1)}$ and $\bar{w}_{cpk}^{(1)}$ are linearly dependent.

6. LINEAR DEPENDENCE OF THE CONJUGATE SOLUTIONS

Using the "normalized" form of representation of solutions in terms of Γ-functions.

$$\tilde{w}_{cpn}^{(1)}(\rho) = \bar{w}_{cpn}^{(1)}(\rho) \left\{ \prod_{k=1}^{4} \Gamma(\mu_{33kn} + 1) \right\}^{-1}$$

as well as the expression

$$\Gamma^{(L_{ij})}(\mu_{ijn} + m + 1) = \prod_{k=1}^{L_{ij}} \Gamma(\mu_{ijkn} + m + 1)$$

we use the "standard" procedure of [1–4, 7] to obtain the linear dependence of two conjugate solutions with respect to the parameter, for integer multipliers $\mu_{33kn} = \mu_{33nk} = m_{33kn} \in \mathbf{N}$

$$\tilde{w}_{cpk}^{(1)}(\rho) = (-1)^{m_{33nk}} (k_w / 4)^{4m_{33nk}} \tilde{w}_{cpn}^{(1)}(\rho) \tag{6.1}$$

Solutions of the second and higher kinds are determined by the generalized Neumann–Weber–Schläfli formula [1–5, 15], which is based on the fact that the solutions are continuous functions of the parameters [12, 14, 15]:

$$\tilde{w}_{cpk}^{(2)}(\rho) = \pi \lim_{\mu_{33nk} \to m_{33nk}} \frac{\tilde{w}_{cpn}^{(1)}(\rho) \cos \mu_{33nk}\pi - (k_w/4)^{-4m_{33nk}} \tilde{w}_{cpk}^{(1)}(\rho)}{\sin \mu_{33nk}\pi} =$$

$$= \left(\frac{\partial \tilde{w}_{cpn}^{(1)}}{\partial \mu_{33nk}} - (-1)^{m_{33nk}} \left(\frac{k_w}{4} \right)^{-4m_{33nk}} \frac{\partial \tilde{w}_{cpk}^{(1)}}{\partial \mu_{33nk}} \right) \Bigg|_{\mu_{33nk} = m_{33nk}} \tag{6.2}$$

Formulae of type (6.2) determine solutions of the second kind for every root pair. Similar representations can be obtained for solutions of the second kind for integer multipliers μ_{44nk} of the second system.

7. SOLUTIONS OF THE SECOND KIND

For the problem under consideration, we determine solutions of the second kind by the formulae specified earlier (see the paper cited on page 963†).

For transverse displacements with $p = 1$ and $c \neq 1$, we have $\nu_{332} = \nu_{333} = 1$, and so $\mu_{3323} = \mu_{3332} = 0$ and the pair of solutions of the first kind $\tilde{w}^{(1)}_{cp3}(\rho)$ and $\tilde{w}^{(1)}_{cp2}(\rho)$ are linearly dependent with coefficient $(-1)^{m_{3323}}(k_w/4)^{4m_{3323}}$. Therefore a solution of the second kind has the form

$$\tilde{w}^{(2)}_{c13} = \left.\left(\frac{\partial \tilde{w}^{(1)}_{c12}}{\partial \mu_{3323}} - (-1)^{m_{3323}}\left(\frac{k_w}{4}\right)^{-4m_{3323}} \frac{\partial \tilde{w}^{(1)}_{c13}}{\partial \mu_{3332}}\right)\right|_{\mu_{3323}=m_{3332}=0} \tag{7.1}$$

where the solutions of the first kind are given by the formulae

$$\tilde{w}^{(1)}_{cll} = \rho \Sigma^{(4)}_{33l}(\rho)$$

For transverse displacements when $c = 1$ and $p = 0$ (in which case $\lambda_{331} = \lambda_{332} = 1$ and $\nu_{331} = \nu_{332} = 2$, $\mu_{333} = \mu_{334} = 0$), while the multipliers are $\mu_{3312} = \mu_{3321} = \mu_{3334} = \mu_{3343} = 0$, we have two pairs of linearly dependent solutions: $\tilde{w}^{(1)}_{c02}$ and $\tilde{w}^{(1)}_{c01}$, with proportionality coefficient $(-1)^{m_{3312}}(k_w/4)^{4m_{3312}}$, and $\tilde{w}^{(1)}_{c04}$ and $\tilde{w}^{(1)}_{c03}$, with coefficient $(-1)^{m_{3334}}(k_w/4)^{4m_{3334}}$. As a result, we obtain solutions of the second kind

$$\tilde{w}^{(2)}_{c02} = \left.\left(\frac{\partial \tilde{w}^{(1)}_{c01}}{\partial \mu_{3312}} - (-1)^{m_{3312}}\left(\frac{k_w}{4}\right)^{-4m_{3312}} \frac{\partial \tilde{w}^{(1)}_{c02}}{\partial \mu_{3321}}\right)\right|_{\mu_{3312}=m_{3321}=0} \tag{7.2}$$

$$\tilde{w}^{(2)}_{c04} = \left.\left(\frac{\partial \tilde{w}^{(1)}_{c03}}{\partial \mu_{3334}} - (-1)^{m_{3334}}\left(\frac{k_w}{4}\right)^{-4m_{3334}} \frac{\partial \tilde{w}^{(1)}_{c04}}{\partial \mu_{3343}}\right)\right|_{\mu_{3334}=m_{3343}=0}$$

where the solutions of the first kind are given by the formulae

$$\tilde{w}^{(1)}_{c0l} = \rho^2 \Sigma^{(4)}_{33l}(\rho), \quad l = 1,2; \quad \tilde{w}^{(1)}_{c0l} = \Sigma^{(4)}_{33l}(\rho), \quad l = 3,4$$

For longitudinal displacements with $p = 0$ and arbitrary c, the characteristic parameters $\nu_{441} = -\nu_{442} = \nu_{443} = -\nu_{444} = \sqrt{c/d}$ depend only on c, and if $\sqrt{c}$ and $\sqrt{c/d}$ are integers, we have two pairs of linearly dependent solutions of the first kind: a first pair $\tilde{u}^{(1)}_{c01}$ and $\tilde{u}^{(1)}_{c02}$, with proportionality coefficient $(-1)^{m_{4412}}(k_u/4)^{2m_{4412}}$), and a second pair $\tilde{v}^{(1)}_{s01}$ and $\tilde{v}^{(1)}_{c02}$, with coefficient $(-1)^{m_{4434}}(k_v/2)^{2m_{4434}}$. Accordingly, the solutions of the second kind have the form

$$\tilde{u}^{(2)}_{c02} = \left.\left(\frac{\partial \tilde{u}^{(1)}_{c01}}{\partial \mu_{4412}} - (-1)^{m_{4412}}\left(\frac{k_u}{2}\right)^{-2m_{4412}} \frac{\partial \tilde{u}^{(1)}_{c02}}{\partial \mu_{4421}}\right)\right|_{\mu_{4412}=m_{4421}=0} \tag{7.3}$$

$$\tilde{v}^{(2)}_{s02} = \left.\left(\frac{\partial \tilde{v}^{(1)}_{s01}}{\partial \mu_{4434}} - (-1)^{m_{4434}}\left(\frac{k_v}{2}\right)^{-2m_{4434}} \frac{\partial \tilde{v}^{(1)}_{s02}}{\partial \mu_{4443}}\right)\right|_{\mu_{4434}=m_{4443}=0}$$

where the solutions of the first kind are expressed as

$$\tilde{u}^{(1)}_{c0l} = \rho^{\pm\sqrt{c}} \Sigma^{(2)}_{44l}(\rho), \quad \tilde{v}^{(1)}_{s0l} = \rho^{\pm\sqrt{c/d}} \Sigma^{(2)}_{44l}(\rho); \quad l = 1,2$$

(here we have used the notation introduced in (5.1)). In special cases ($c = 1$) and also when $p = 0$, 1 and $c \neq 1$, these solutions are identical with known solutions [5–7, 9] and are expressed by cylindrical Bessel functions.

When $k_u = k_v = k_u = 0$, i.e., there is no resistance of the medium, the series in the structure of the solutions vanish and the solutions become those already known [9, 10, 13] for the longitudinal-transverse

†See also GRIGOLYUK, E. I., KOROL, Ye. Z., IZMAILOVA, M. Ye. and VOZNESENSKAYA, M. Ye., Bending of thin cylindrically orthotropic circular plates on an elastic base. Inst. Mekhaniki Mosk. Gos. Univ., Moscow, 1997. Dep. At VINITI 07.10.97, No. 2974–B97.

bending of orthotropic circular plates; the case $p = 0$ is that of axisymmetric loading while $c = 1$ is for an isotropic material.

8. PARTICULAR SOLUTIONS OF THE INHOMOGENEOUS BENDING EQUATION

Particular solutions of the inhomogeneous bending equation will be determined by Lagrange's method, in the form

$$U_{cp}(\rho) = \sum_{l=1}^{4} \mathbf{A}_{cpl}(\rho)\tilde{u}_{cpl}(\rho), \quad V_{sp}(\rho) = \sum_{l=1}^{4} \mathbf{A}_{cpl}(\rho)\tilde{\upsilon}_{spl}(\rho) \tag{8.1}$$

$$W_{cp}(\rho) = \sum_{l=1}^{4} \mathbf{B}_{cpl}(\rho)\tilde{w}_{cpl}(\rho),$$

so that we need to determine the functions $\mathbf{A}_{cpl}(\rho)$ and $\mathbf{B}_{cpl}(\rho)$.

For inhomogeneous equations of Nth order with variable coefficients, given a known system of fundamental solutions $\tilde{w}_n(\rho)$, a particular solution may be expressed as [15, 16]

$$W(\rho) = \sum_{n=1}^{N} \tilde{w}_n(\rho)\int \frac{\mathbf{V}_{N_n}(\rho)}{\mathbf{V}_N(\rho)} f(\rho)d(\rho) \tag{8.2}$$

where

$$\mathbf{V}_N(\rho) = \mathbf{V}_N\{\tilde{w}_n\} = \det\left\| \frac{d^k\tilde{w}_l(\rho)}{d\rho^k} \right\|, \quad k = 0,1,2,3; \quad n = 1,2,3,4$$

$\mathbf{V}_N(\rho)$ being the Wronskian and $\mathbf{V}_{Nn(\rho)}$ the cofactor of the determinant at the intersection of the fourth row and the nth column, obtained by replacing the latter column by a column of free terms $\{0, 0, 0, \rho^4 f(\rho)\}$. By the Liouville–Ostrogradskii formula, the Wronskian for the equations of longitudinal-transverse bending under consideration is $\mathbf{V}_4(\rho) = \rho^{-2}$.

For the components of displacement, say, under axisymmetric loading $U_c(\rho)$, $V_s(\rho)$ in the form (8.2), $\mathbf{V}_2(\rho)$ is the determinant of the matrices $\|\bar{u}_{0n}\|$ or $\|\upsilon_{0n}\|$, $n = 1, 2$, and $\mathbf{V}_{2n}(\rho)$ are the cofactors of the corresponding determinants at the intersection of the second row and the nth column, obtained by replacing the nth column by the column of free terms $(0, f_1(\rho))$ or $(0, f_2(\rho))$. The Wronskian in this case is $\mathbf{V}_2(\rho) = \rho^{-1}$, and the solution is

$$U_0(\rho) = \frac{\bar{u}_{01}^{(1)}(\rho)}{\mathrm{v}_{111} - \mathrm{v}_{112}}\int \bar{u}_{02}^{(1)}(\rho)\rho f_{10}(\rho)d\rho + \frac{\bar{u}_{02}^{(1)}(\rho)}{\mathrm{v}_{112} - \mathrm{v}_{111}}\int \bar{u}_{01}^{(1)}(\rho)\rho f_{10}(\rho)d\rho \tag{8.3}$$

$$V_0(\rho) = \frac{\bar{\upsilon}_{01}^{(1)}(\rho)}{\mathrm{v}_{221} - \mathrm{v}_{222}}\int \bar{\upsilon}_{02}^{(1)}(\rho)\rho f_{20}(\rho)d\rho + \frac{\bar{\upsilon}_{02}^{(1)}(\rho)}{\mathrm{v}_{222} - \mathrm{v}_{221}}\int \bar{\upsilon}_{01}^{(1)}(\rho)\rho f_{20}(\rho)d\rho.$$

If there is no reaction of the base, $k_u = k_\upsilon = k_w = 0$, these solutions are identical with the known ones [5–7, 9, 13].

9. INTEGRATION CONSTANTS A_{CPL} AND B_{CPL}. SOFTWARE PACKAGE AND COMPUTED RESULTS

The values of the coefficients A_{cpl} and B_{cpl} are determined from boundary conditions of various types (kinematic and force), represented by the components of the Fourier expansion:

displacements and deflection at the outer contour ($\rho = 1$) or inner contour ($\rho = \beta$):

$$(u, w)_{cp}(\rho = 1, \beta) = (u, w)_{cp1,\beta}, \quad \upsilon_{sp}(\rho = 1, \beta) = \upsilon_{sp1,\beta} \tag{9.1}$$

angle of rotation of the normal $\vartheta = \partial w(\rho,\theta)/\partial\rho = w'_p(\rho,\theta)$:

$$\vartheta_{1cp}(\rho=1,\beta)=w'_{cp}(\rho=1,\beta)=w'_{cp1,\beta} \tag{9.2}$$

stresses and moments:

$$(n,m,r)_{1cp}(\rho=1,\beta)=(n,m,r)_{1cp1},\ \ (s,h)_{1sp}(\rho=1,\beta)=(s,h)_{1sp1,\beta} \tag{9.3}$$

elastic imbedding at the outer contour ($\rho = 1$) or contour inner ($\rho = \beta$):

$$(n,m,r)_{1cp}(\rho)=\Lambda_{(n,m,r)}(u,\vartheta,w)_{cp}(\rho),\ \ (s,h)_{1sp}(\rho)=\Lambda_{(s,h)}(\upsilon,\vartheta)_{sp}(\rho) \tag{9.4}$$

$$\Lambda_{(n,s,m,r,h)}=\begin{cases}\Lambda_{1(n,s,m,r,h)} & \text{for}\ \ \rho=1\\ \Lambda_{\beta(n,s,m,r,h)} & \text{for}\ \ \rho=\beta\end{cases}$$

for a solid plate ($\beta = 0$), it is stipulated that at the centre ($\rho = 0$) the radial, circumferential and transverse displacements, $u(0, \theta)$, $v(0, \theta)$ and $w(0, \theta)$, respectively, should be bounded.

The coefficients A_{cpl} and B_{cpl} (l = 1, 2, 3, 4), must satisfy a fourth-order system of algebraic equations

$$\|a_{pkl}\|\ \ \|A_{cpl}\|=\|b_{pk}\|,\ \ \|g_{pkl}\|\ \ \|B_{cpl}\|=\|d_{pk}\| \tag{9.5}$$

where a_{pkl} and g_{pkl} are the values of expressions (9.1)–(9.4), and b_{pk} and d_{pk} are the boundary values of the functions $(u, v, w, n_1, s_1, r_1)p$. In formulating the boundary conditions and computing the characteristics of the stressed state the normalized quantities.

$$n_{(1,2)}(\rho,\theta)=\frac{N_{(1,2)}(\rho,\theta)}{B_1},\ \ m_{(1,2)}(\rho,\theta)=\frac{M_{(1,2)}(\rho,\theta)a^2}{D_1 h}$$

$$r_{(1,2)}(\rho,\theta)=\frac{R_{(1,2)}(\rho,\theta)a^3}{D_1 h},\ \ h_1(\rho,\theta)=\frac{H(\rho,\theta)a^3}{D_3 h},\ \ s_1(\rho,\theta)=\frac{S(\rho,\theta)}{B_3}$$

were determined from the thermo-elasticity relations

$$n_1(\rho,\theta)=\frac{\partial u}{\partial\rho}+\omega_{21}\left(\frac{u}{\rho}+\frac{1}{\rho}\frac{\partial\upsilon}{\partial\theta}\right)-(\alpha_1+\omega_{21}\alpha_2)n_T(\rho,\theta)$$

$$s_1(\rho,\theta)=\frac{1}{2}\left(\frac{\partial\upsilon}{\partial\rho}-\frac{\upsilon}{\rho}+\frac{1}{\rho}\frac{\partial u}{\partial\theta}\right),\ \ h_1(\rho,\theta)=-\frac{1}{\rho}\frac{\partial^2 w}{\partial\rho\partial\theta}-\frac{1}{\rho^2}\frac{\partial w}{\partial\theta}$$

$$m_1(\rho,\theta)=-\frac{\partial^2 w}{\partial\rho^2}-\omega_{21}\left(\frac{1}{\rho}\frac{\partial w}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 w}{\partial\theta^2}\right)-\frac{a^2}{h^2}(\alpha_1+\omega_{21}\alpha_2)m_T(\rho,\theta)$$

$$r_1(\rho,\theta)=-\frac{\partial^3 w}{\partial\rho^3}-\frac{1}{\rho}\frac{\partial^2 w}{\partial\rho^2}+\frac{c}{\rho^2}\frac{\partial w}{\partial\rho}+\frac{\omega_{21}+c}{\rho^3}\frac{\partial^2 w}{\partial\theta^2}-\frac{\omega_{21}}{\rho^2}\frac{\partial^3 w}{\partial\rho\partial\theta^2}-$$

$$-\frac{a^3}{h^3}\left((\alpha_1+\omega_{21}\alpha_2)\frac{\partial m_T(\rho,\theta)}{\partial\rho}+\left[(\alpha_1+\omega_{21}\alpha_2)-c(\omega_{12}\alpha_1+\alpha_2)\right]\frac{m_T(\rho,\theta)}{\rho}\right)$$

$$n_2(\rho,\theta)=c\left[\omega_{12}\frac{\partial u}{\partial\rho}+\frac{u}{\rho}+\frac{1}{\rho}\frac{\partial\upsilon}{\partial\theta}\right]-(\omega_{12}\alpha_1+\alpha_2)n_T(\rho,\theta)$$

$$m_2(\rho,\theta)=-c\left[\omega_{12}\frac{\partial^2 w}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial w}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 w}{\partial\theta^2}\right]-\frac{a^2}{h^2}(\omega_{12}\alpha_1+\alpha_2)m_T(\rho,\theta)$$

$$r_2(\rho,\theta)=-\frac{\omega_{21}}{\rho}\frac{\partial^3 w}{\partial\rho^2\partial\theta}-\frac{c+4d}{\rho^2}\frac{\partial^2 w}{\partial\rho\partial\theta}-\frac{4d}{\rho^3}\frac{dw}{\partial\theta}-\frac{c}{\rho^3}\frac{\partial^3 w}{\partial\theta^3}+\frac{a^3}{h^3}c(\omega_{12}\alpha_1+\alpha_2)\frac{1}{\rho}\frac{\partial m_T(\rho,\theta)}{\partial\theta}$$

In analysing the stress–strain state of plates from the thermomechanical and geometrical parameters, use was made of the "self-similar solutions" obtained, where the number of parameters was reduced to a minimum. The following values were taken in the computations

$$\omega_{12}\omega_{21} = \omega_0^2 = 0.0144, \quad \omega_{12} = \frac{\omega_0}{\sqrt{c}}, \quad \omega_{21} = \omega_0\sqrt{c}$$

$$\frac{2G_{12}}{E_1} = \frac{\sqrt{c}}{1+\omega_0}, \quad E_2 = cE_1, \quad \beta = \frac{1}{2}, \quad \frac{a}{h} = 10$$

$$\alpha_1^2 + 2\omega_0\alpha_1\alpha_2 + c\alpha_2^2 = \text{const}, \quad \alpha_1 = \sqrt{c}\alpha_2$$

$$k_w / \sqrt{2} = \{0, 1/4, 1/2, 1, 2, 4, 8\}$$

Figures 1–3 present the computational characteristics of a disk, hinge-supported at the outer contour and loaded with a radial uniformly distributed bending moment at the inner contour. The computations showed that it was sufficient to consider only 10–15 terms in expansions (3.1), in order that any further increase in the number of terms should affect the result by less than a few hundredths of one percent when the transverse and longitudinal stresses and bending moments, i.e. the derivatives of the unknown functions – the components of the displacements – were computed. As is evident from (5.1) and (5.2), when $k_w = k_u k_v = 0$ (no resistance of the base), these solutions are identical with those of the problem of longitudinal-transverse bending of circular anisotropic plates [9, 12] subject to non-axisymmetric loading and heating. The presence of factors of the type $\rho^{\nu_l - k}$ $(k = 1, 2, 3)$ indicates a special feature of the dependence of the stresses and moments in the neighbourhood of $\rho = 0$ on the anisotropy parameter c: when $c < 1$ they increase rapidly, but when $c > 1$ they tend to zero [11, 13]. When $c = 1$ (an isotropic plate on an elastic base) solutions (5.1) become those known from [5–10], expressed in terms of second-order cylindrical functions – Bassel functions. Analysis of the graphs (Figs 1–3) for a constant anisotropy coefficient ($c = 16.8$) points to the significant dependence, first, of the profile of the bent plate (deflection $\bar{w} = w/h$) on the coefficient of elasticity k_w of the base (Fig. 1): as the resistance of the case is increased, the effect of the load (in this case – of the radial bending moment) is localized; changes occur in the distributions curves of the radial bending moment $\bar{m}_1 = M_1/M_0$ and the circumferential moment $m_2 = M_2/M_0$ reduced relative to the moment M_0 acting at the inner contour (Fig. 2), and in that of the transverse stress $\bar{r}_1 = R_1 a/M_0$ reduced relative to M_0/a (Fig. 3).

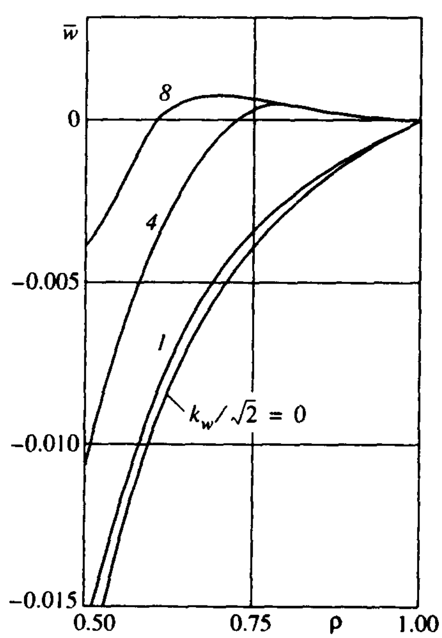

Fig. 1

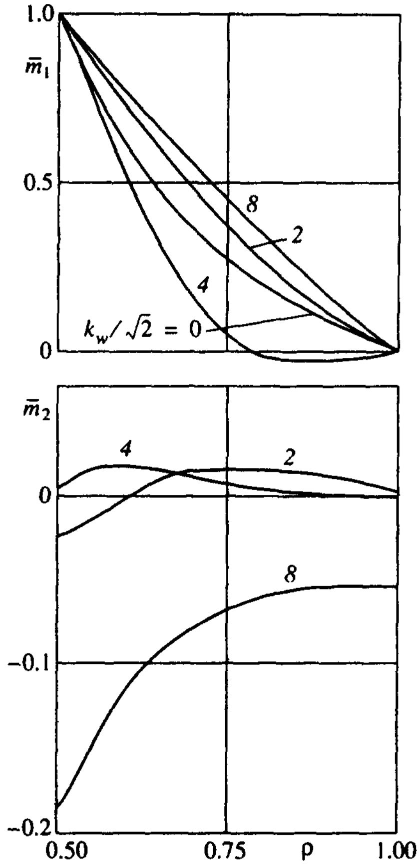
1.0
$\bar{m}_1$
0.5
0
8
2
4
$k_w/\sqrt{2}$ = 0
$\bar{m}_2$
4
2
0
8
−0.1
−0.2
0.50
0.75
ρ
1.00

Fig. 2

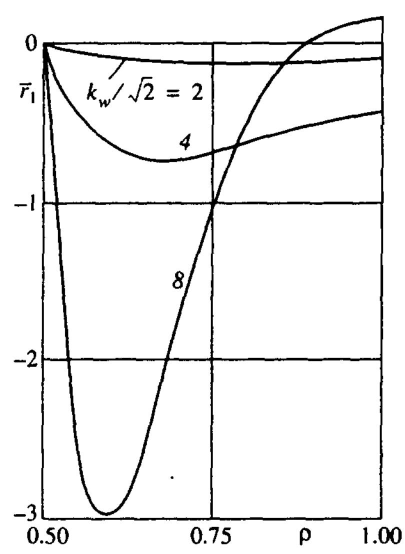
0
$\bar{r}_1$
$k_w/\sqrt{2}$ = 2
4
−1
8
−2
−3
0.50
0.75
ρ
1.00

Fig. 3

REFERENCES

1. WHITTAKER, E. T. and WATSON, G. N., *A Course of Modern Analysis*. Cambridge University Press, Cambridge, 1927.
2. WEBER, H., Über eine Darstellung willkürlicher Funktionen durch Bessel' schen Funktionen. *Math. Ann.*, 1873, **6**, 146–161.
3. SCHLÄFLI, L., Sull'uso delle linee lungo le quali il valore assoluto di una funzione e costante. *Ann. Math.*, 1873–1875, **6**, 1–20.
4. NEUMANN, K., *Theorie der Bessel'schen Funktionen*. Teubner, Leipzig, 1867.
5. LOMMEL, E. L., Integration der Gleichung $x^{m+1/2}\frac{\delta^{2m+1}y}{\delta x^{2m+1}} \pm y = 0$ durch Bessel'sche Funktionen. *Math. Ann.*, 1870, 2, 624–635.
6. DINNIK, A. N., *Collected Papers*, Vol. 2: *Applications of Bessel Functions to Problems of Elasticity Theory*. Izd. Akad. Nauk UkrSSR, Kiev, 1955.
7. KORENEV, B. G., *Introduction to the Theory of Bessel Functions*. Nauka, Moscow, 1971.
8. GRIGOLYUK, E. I. and MAGERRAMOVA, L. A., The stability of circular homogeneous and inhomogeneous plates. *Izd. Akad. Nauk SSSR. MTT*, 1981, 2, 111–138.
9. GRIGOLYUK, E. I., *The strength, Vibrations and Stability of Circular Plates*, Pt 1. Inst. Mekhaniki Mosk. Gos. Univ., Moscow, 1997.
10. VAINBERG, D. V. and VAINBERG, Ye. D., *Plates and Disks, Beams-Walls*. Gosstroiizdat, Kiev, 1959.
11. KOROL, Ye. Z., Boundary-value problems of the bending of cylindrically orthotropic circular plates on an elastic inhomogeneous base. *Proceedings of the 4th International Conference on the Mechanics of Inhomogeneous Structures*. Ternopol, 1995, 165–166.
12. KOROL', Ye. Z., The computation of orthotropic circular plates subject to longitudinal-transverse bending. *Problemy Mashinostroyeniya i Nadezhnosti Mashin*, 2000, 2, 62–70.
13. KOROL', Ye. Z., The phenomenology of the thermomechanics of reversible processes of anisotropic deformable media. *Proceeding of the Jubilee International Symposium on "Current Problems of Mechanics of Continuous Fnable Media"*. Mosk. Aviatz, Inst., Moscow, 1997, p. 105.
14. CODDINGTON, E. A. and LEVINSON, N., *Theory of Ordinary Differential Equations*. McGraw-Hill, New York, 1955.
15. INCE, E. L., *Integration of Ordinary Differential Equations*. Longmans and Green, London, 1927.

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